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# On Topological Shadowing and Chain Properties of IFS on Uniform Spaces

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## Abstract

We define notions such as pseudo-orbit, topological shadowing, and topological chain transitivity of iterated function systems on compact uniform spaces. We prove that these notions are invariant under topological conjugacy on a compact uniform space. For an *IFS* on a compact uniform space with topological shadowing property, we show that the topological chain transitivity implies topological transitivity. We also show that in a connected compact uniform space, notions such as topological chain mixing, totally topological chain transitive, topological chain transitive, and topological chain recurrent are equivalent.

**Keywords:** uniform space; shadowing; mixing; iterated function systems (*IFS*); transitivity.

# 1 Introduction

This paper aims to extend the concepts such as chain transitivity, shadowing, chain mixing, and chain recurrence of iterated function systems to the case when the phase space is a uniform space. Iterated function systems(*IFS*s) were first proposed by Hutchinson in [12] and popularized by Barnsley and Demko in [7]. Recently, Samuel and Tetenov [17] discuss the notions of attractors of *IFS* on uniform spaces. Essential ideas of a dynamical system such as minimality, transitivity, attractors, shadowing could be extended to *IFS*. Bahabadi [6] extended the notion of shadowing to iterated function systems. The shadowing property, which has been introduced independently by Anosov [5] and Bowen [8], is motivated by computer simulations. It plays a crucial part in the general qualitative theory of dynamical systems. In a system with shadowing property, the actual orbits follow the pseudo orbits. Pseudo orbits are estimated orbits obtained from the computer simulations of dynamical systems. They are more valuable than the actual ones as they could detect many dynamical concepts such as mixing and recurrence that actual orbits could not obtain. The notion of shadowing helps us determine the differences between actual and approximate solutions on infinite time intervals and understand error terms' influence. Also, a system with shadowing property satisfies many other dynamical notions. For example, chain transitive dynamical systems with shadowing properties are topologically transitive on compact metric spaces. Therefore, studying the shadowing property and extending its study to iterated function systems on general topological spaces will be interesting.

Generally, one examines the notions for dynamical systems on compact metric spaces. However, recently, many authors have studied dynamical systems in non-metrizable topological spaces. Alcaraz and Sanchis [3], first extended the study of dynamical systems to the case when the phase space is the completion of totally bounded uniform space. They proved that the notions of minimality, transitivity, and chaos (in Devaney's sense) could be extended to uniform spaces. Das et al. [10] defined the concepts of expansivity, topological chain recurrence, and topological shadowing for homeomorphisms. On a non-metrizable and non-compact space, Ahmadi [1] introduced the concepts of chain transitivity, ergodic shadowing, and topological ergodicity for a dynamical system. Das and Das [9], define the topological ergodic shadowing, topological $\underline{d}$ -shadowing, topological pseudo orbital specification, and topological weak specification of a continuous map over a uniform space. They showed that these notions are equivalent for a system with a uniformly continuous map having topological shadowing property over a totally bounded uniform space. Recently, Ahmadi et al. [2] proved that the topological chain mixing, totally topological chain transitive, topological chain transitive, and topological chain recurrent properties of a dynamical system are equivalent on a connected compact uniform space. Devi and Mangang [11] have also studied the notions of positive expansivity, chain transitivity, rigidity, and specification on general topological spaces. Further, Wong and Salleh [19] have extended the notions of sensitivity, transitivity, and mixing to the set-valued dynamical systems.

Motivated by the various works on the extension of the notions in dynamical systems to the case when the phase space is a uniform space, we want to extend the various notions of a dynamical system to iterated function systems when the phase space is a compact uniform space. The class of compact metric spaces is a subclass of the class of compact uniform spaces, so the research work will generate interests of wider applications. Throughout the paper, we consider the *IFS* $\mathfrak{T} = \{T; g_\lambda | \lambda \in \Lambda\}$ to be a collection of uniformly continuous self maps on a compact uniform space $T$ and $\Lambda$ a non-empty finite set.

The paper is arranged as follows. In section 2, we give some preliminaries on dynamical systems, iterated function systems and uniform spaces on uniform spaces. In section 3, we define pseudo orbit and topological shadowing property(*TSP*)of iterated function systems on uniform spaces. In Theorem 2.1, we show that the topological shadowing property is invariant under a topological conjugacy. We give an example of an *IFS* with *TSP* in Example 3.1. In Theorem 3.2, we give a sufficient condition for an *IFS* on a compact uniform space to have topological shadowing property. In Theorem 3.3, we show that if an *IFS* $\mathfrak{T}$ on a compact uniform space has *TSP*, then so does $\mathfrak{T}^n$ for every $n > 1$. In section 4, we study the topological chain properties of iterated function systems on uniform spaces. We define topological chain transitive($TCT$), totally topological chain transitive($TTCT$), topological chain mixing($TCM$), and topological chain recurrence($TCR$) of iterated function systems on uniform spaces. In Theorem 4.1 and Corollary 4.2, we show respectively that the topological chain transitive and topological chain mixing properties of iterated function systems are invariant under a topological conjugacy on a compact uniform space. We show that topological chain transitive *IFS*s having topological shadowing property are topological transitive on compact uniform spaces in Theorem 4.3. We also show that an *IFS* having topological shadowing property is topological mixing iff it is topological chain mixing on compact uniform spaces in Theorem 4.4. In Theorem 4.7, we show that the topological chain mixing, totally topological chain transitive, topological chain transitive, and topological chain recurrent properties of an *IFS* on a connected compact uniform space are equivalent. Lastly, in section 5, we give a conclusion of our findings.

## 2 Preliminaries

In this section, we give some preliminaries of dynamical systems, iterated function systems and uniform spaces which are to be used in the paper. First, we define some notions of a dynamical system $(T, g)$, where $T$ is a compact metric space with metric $d$, and $g$ a continuous map from $T$ into $T$. Let $\mathbb{Z}_+$ represent the set of whole numbers. Let $O_a(g) = \{g^n(a) : n \in \mathbb{Z}_+\}$ denote the orbit of a point $a \in T$ under the dynamical system $(T, g)$. Equivalently, any sequence $\{a_i\}_{i\in\mathbb{Z}_+}$ in $T$ is an orbit if $g(a_i) = a_{i+1},\ \forall\ i \geq 0$. A point $a \in T$ is considered to be a periodic point of $g$ if $g^p(a) = a$ for some $p > 0$ and $g^i(a) \neq a$ for all $1 \leq i \leq p - 1$. Given $\delta > 0$, a finite sequence $\{a_i\}_{i=0}^{n}$ is known as a $\delta$-chain of length $n$ if $d(g(a_i), a_{i+1}) < \delta, \forall\, i \leq n - 1$. An infinite $\delta$-chain is considered to be a $\delta$-pseudo orbit. A dynamical system $(T, g)$ is considered to have shadowing property if for a given $\epsilon > 0$, there exists $\delta > 0$ such that for any $\delta$-pseudo orbit $\{a_i\}_{i\in\mathbb{Z}_+}$, there exists an orbit $\{b_i\}_{i\in\mathbb{Z}_+}$ such that $d(b_i, a_i) < \epsilon, \forall\, i \in \mathbb{Z}_+$.

A dynamical system $(T, g)$ is said to be

1. *topologically transitive* if for any pair of non-empty open sets $M, N \subset T$, there is some $n \geq 0$ such that $g^n(M) \cap N \neq \phi$.
2. *topologically mixing* if for any pair of non-empty open sets $M, N \subset T$, there is $P > 0$ such that $g^n(M) \cap N \neq \phi, \forall\, n \geq P$.
3. *chain transitive* if for every $\delta > 0$ and for any $a, b \in T$, there exists a $\delta$-chain from $a$ to $b$.
4. *chain mixing* if for every $\delta > 0$ and for any $a, b \in T$, there exists $P > 0$ such that $\forall\, n \geq P$, there is a $\delta$-chain of length $n$ from $a$ to $b$.
5. *chain recurrent* if for every $\delta > 0$ and for any $a \in T$, there exists a $\delta$-chain from $a$ to itself.

An *IFS*, $\mathfrak{T} = \{T; g_\lambda | \lambda \in \Lambda\}$ is a collection of continuous functions $g_\lambda : T \to T$, $\lambda \in \Lambda$, where $\Lambda$ is a non-empty finite set and $(T, d)$ is a compact metric space. We denote $\Lambda^{\mathbb{Z}_+}$ to be the set of all sequences $\{\lambda_i\}_{i\in\mathbb{Z}_+}$ of symbols in $\Lambda$. Thus, we use the short notation $\mathfrak{T}_{\sigma_i} = g_{\lambda_{i-1}} \circ \cdots \circ g_{\lambda_1} \circ g_{\lambda_0}$. Let $\sigma = \{\lambda_0, \lambda_1, \lambda_2, \cdots\}$ be a typical element of $\Lambda^{\mathbb{Z}_+}$. A sequence $\{a_i\}_{i\in\mathbb{Z}_+}$ in $T$ is said to be an orbit of an iterated function system $\mathfrak{T}$ if there exists $\sigma \in \Lambda^{\mathbb{Z}_+}$, such that $a_i = \mathfrak{T}_{\sigma_i}(a_0)$ for any $i \in \mathbb{Z}_+$, where $\mathfrak{T}_{\sigma_i}(a_0) = g_{\lambda_{i-1}} \circ \cdots \circ g_{\lambda_1} \circ g_{\lambda_0}(a_0)$ and $\mathfrak{T}_{\sigma_0}(a_0) = a_0$. Thus, for every $\sigma \in \Lambda^{\mathbb{Z}_+}$, orbit of a point $a \in T$ is defined as $O_\sigma(a) = \{\mathfrak{T}_{\sigma_i}(a) : i \in \mathbb{Z}_+\}$. Let $\mathfrak{T}$ be an iterated function system, for a fixed positive integer $n$, we define $\Lambda^n = \{(\lambda_0, \lambda_1, \cdots, \lambda_{n-1}) : \lambda_i \in \Lambda,\ i \in \{0, 1, \cdots, n-1\}\}$; $G_\mu = g_{\lambda_{n-1}} \circ \cdots \circ g_{\lambda_1} \circ g_{\lambda_0}$; and $\mathfrak{T}^n = \{G_\mu \mid \mu \in \Lambda^n\}$.

Fatehi Nia [15] introduced the following definitions of topological transitive, topological mixing, and totally topological transitive iterated function systems. He also introduced the notion of topological conjugacy of iterated function systems.

An IFS $\mathfrak{T} = \{T; g_\lambda | \lambda \in \Lambda\}$ is considered to be

1. *topological transitive*(*TT*) if for any pair of non-empty open sets $M, N \subset T$, there is $\sigma \in \Lambda^{\mathbb{Z}_+}$ such that $\mathfrak{T}_{\sigma_n}(M) \cap N \neq \phi$ for some $n \geq 0$.
2. *topological mixing*(*TM*) if for every pair of non-empty open sets $M, N \subset T$, there is $P > 0$ such that $\mathfrak{T}_{\sigma_n}(M) \cap N \neq \phi, \forall\, n \geq P$ and for some $\sigma \in \Lambda^{\mathbb{Z}_+}$.
3. *topologically conjugate* to an iterated function system $\mathfrak{R} = \{R; s_\lambda | \lambda \in \Lambda\}$ if there exists a homeomorphism $h : T \to R$ for which $s_\lambda = h \circ g_\lambda \circ h^{-1}$, for all $\lambda \in \Lambda$. In such case, $h$ is said to be a topological conjugacy.
4. *totally topological transitive* if $\mathfrak{T}^n$ is topological transitive for all $n > 0$.

Uniform space was developed by Andre Weil in [18]. By a uniform space, we refer to a pair $(T, \mathcal{U})$, where $T$ is a non-empty set and $\mathcal{U}$ is a non-empty family of subsets of $T \times T$ satisfying the following conditions:

1. Every member $A$ of $\mathcal{U}$ includes the diagonal $\Delta$, where $\Delta = \{(a, a) \in T \times T : a \in T\}$.
2. $A^{-1} \in \mathcal{U}$ if $A \in \mathcal{U}$, where $A^{-1} = \{(b, a) \in T \times T : (a, b) \in A\}$.
3. For every $A \in \mathcal{U}$, there is some $B \in \mathcal{U}$ for which $B \circ B \subset A$ where $B \circ B = \{(a, b) | \exists\, c \in T$ such that $(a, c) \in B$ and $(c, b) \in B\}$.
4. $A, B \in \mathcal{U}$, then $A \cap B \in \mathcal{U}$.
5. $A \in \mathcal{U}$ and $A \subset B \subset T \times T$, then $B \in \mathcal{U}$.

$\mathcal{U}$ is said to be a uniformity on $T$. The members of $\mathcal{U}$ are called entourages. $A \subset T \times T$ is considered to be symmetric if $A = A^{-1}$. For every entourage $A \in \mathcal{U}$, we can find a symmetric entourage $B \in \mathcal{U}$ for which $B \circ B \subset A$. The uniform topology $\mathcal{T}_u$ is given by $\mathcal{T}_u = \{M \subset T :$ for every $a \in M$ there is $A \in \mathcal{U}$ with $A[a] \subset M\}$, where $A[a] = \{b \in T : (a, b) \in A\}$ is known as the cross section of $A$ at $a$. We consider a topological space $T$ to be uniformizable if there is a uniformity on $T$ such that the uniform topology is the given topology. A topological space is considered to be uniformizable if and only if it is completely regular. Let $(T, \mathcal{U})$ and $(R, \mathcal{V})$ be two uniform spaces. A function $g : T \to R$ is considered to be uniformly continuous if for every $A \in \mathcal{V}$, there is some $B \in \mathcal{U}$ such that $(g(a), g(b)) \in A$ whenever $(a, b) \in B$. Throughout the paper, we use the notation $B^n$ to denote $\underbrace{B \circ B \circ \cdots \circ B}_{n\ times}$, where $n$ is a positive integer.

## 3 Topological Shadowing Property of Iterated Function Systems

In this section, we define pseudo-orbit and topological shadowing property of iterated function systems on uniform spaces. We give an example of an iterated function system with topological shadowing property. We show that the topological shadowing property of iterated function systems is invariant under a topological conjugacy. We also give a sufficient condition for an *IFS* on a compact uniform space to have topological shadowing property.

**Definition 3.1.** *Let $\mathfrak{T}$ be an IFS over a uniform space $(T,\mathcal{U})$, and $A \in \mathcal{U}$. An infinite sequence $\{a_i\}_{i\in\mathbb{Z}_+}$ in $T$ is said to be an $A$-pseudo orbit if $\exists\ \sigma \in \Lambda^{\mathbb{Z}_+}$ such that for every $\lambda_i \in \sigma$, we have $(g_{\lambda_i}(a_i), a_{i+1}) \in A$, $\forall\ i \in \mathbb{Z}_+$.*

**Definition 3.2.** *Let $\mathfrak{T}$ be an IFS on a uniform space $(T,\mathcal{U})$ and $A, B \in \mathcal{U}$. A $B$-pseudo orbit $\{b_i\}_{i\in\mathbb{Z}_+}$ of $T$ is said to be $A$-shadowed by some point $a \in T$ if $\exists\ \sigma \in \Lambda^{\mathbb{Z}_+}$ such that $(\mathfrak{T}_{\sigma_i}(a), b_i) \in B\ \ \forall\ i \in \mathbb{Z}_+$.*

**Definition 3.3.** *An IFS $\mathfrak{T}$ on a uniform space $(T,\mathcal{U})$ is said to have topological shadowing property$(TSP)$ if for every entourage $A \in \mathcal{U}$, there exists an entourage $B \in \mathcal{U}$ such that every $B$-pseudo orbit is $A$-shadowed by some point in $T$.*

Fatehi Nia[15] confirmed that shadowing is invariant under a topological conjugacy and that $\mathfrak{T}^n$ have shadowing property if $\mathfrak{T}$ have shadowing property. He provides sufficient conditions for an *IFS* to have shadowing property. Theorems 3.1, 3.2, 3.3 and Corollary 3.1 are the extensions of these results in compact uniform spaces.

**Theorem 3.1.** *If $\mathfrak{T} = \{T; g_\lambda|\lambda \in \Lambda\}$ and $\mathfrak{R} = \{R; s_\lambda|\lambda \in \Lambda\}$ are two conjugated IFSs on compact uniform spaces $(T,\mathcal{U})$ and $(R,\mathcal{V})$ respectively, where $\Lambda$ is a non-empty finite set, then $\mathfrak{T}$ has TSP iff $\mathfrak{R}$ does.*

*Proof.* Take $h : T \to R$ to be a topological conjugacy. Suppose $A \in \mathcal{V}$. Given that $T$ is a compact uniform space, then $h$ is uniformly continuous. Thus, there is an entourage $M \in \mathcal{U}$ satisfying

$$(a, b) \in M \implies (h(a), h(b)) \in A. \tag{1}$$

First, suppose that $\mathfrak{T}$ has the *TSP*. Since $M \in \mathcal{U}$, there is an entourage $N \in \mathcal{U}$ such that any $N$-pseudo orbit $\{a_i\}_{i\in\mathbb{Z}_+}$ in $T$ is $M$-shadowed by some point in $T$. Since $h^{-1}$ is uniformly continuous, there exists $B \in \mathcal{V}$ such that

$$(u, v) \in B \implies (h^{-1}(u), h^{-1}(v)) \in N. \tag{2}$$

Let $\{b_i\}_{i\in\mathbb{Z}_+}$ be a $B$-pseudo orbit in $R$. Then, there is some $\sigma \in \Lambda^{\mathbb{Z}_+}$ so that for any $\lambda_i \in \sigma$, $(s_{\lambda_i}(b_i), b_{i+1}) \in B$.
By (2), we have $(h^{-1}(s_{\lambda_i}(b_i)), h^{-1}(b_{i+1})) \in N$. Put $a_i = h^{-1}(b_i)$, then $(h^{-1}(s_{\lambda_i}(h(a_i))), a_{i+1}) \in N$.
Thus, $(g_{\lambda_i}(a_i), a_{i+1}) \in N$. Hence, $\{a_i\}_{i\in\mathbb{Z}_+}$ is a $N$-pseudo orbit of $T$.
Therefore, there is a point $c$ in $T$ so that $(\mathfrak{T}_{\sigma_i}(c), a_i) \in M,\ \forall\ i \in \mathbb{Z}_+$.
By (1) we have $(h(\mathfrak{T}_{\sigma_i}(c)), h(a_i)) \in A$, which implies $(\mathfrak{R}_{\sigma_i}(h(c)), b_i) \in A$.
Thus, the $B$-pseudo orbit $\{b_i\}_{i\in\mathbb{Z}_+}$ of $R$ is $A$-shadowed by a point $d = h(c)$ in $R$.
Hence, $\mathfrak{R}$ has *TSP*. The converse can be proved in similar way.
Hence, $\mathfrak{T}$ has *TSP* iff $\mathfrak{R}$ does. □

The following is an example of *TSP* in uniform space.

**Example 3.1.** *Let $T = [0, 1]$ be the closed unit interval with the usual metric. Take $\Lambda = \{\lambda_1, \lambda_2\}$, and let $g_{\lambda_1}, g_{\lambda_2} : T \to T$ be defined by $g_{\lambda_1}(a) = \frac{a}{2}$ and $g_{\lambda_2}(a) = \frac{a}{3}$. Then, the IFS $\mathfrak{T} = \{T; g_\lambda | \lambda \in \Lambda\}$ has TSP.*

*Proof.* For a given $\epsilon > 0$, put $A_\epsilon = \{(a, b) : |a-b| < \epsilon\}$. $A_\epsilon$ is an entourage of the unique uniformity of $T$. Set $\delta = \frac{\epsilon}{2}$, then $A_\delta = \{(a, b) : |a - b| < \delta\}$ is also an entourage. Let $\{a_0, a_1, a_2, \cdots\}$ be a $A_\delta$-pseudo orbit corresponding to the infinite sequence $\sigma = \{\lambda_i\}_{i=0}^{\infty}$, where $\lambda_i = \begin{cases} \lambda_1, & \text{if } i \text{ is even,} \\ \lambda_2, & \text{if } i \text{ is odd.} \end{cases}$

If $i$ is even,

$$\left| \frac{a_i}{2} - a_{i+1} \right| < \delta.$$

If $i$ is odd,

$$\left| \frac{a_i}{3} - a_{i+1} \right| < \delta.$$

By induction, we claim that $a = a_0$ $A_\epsilon$-shadows the $A_\delta$-pseudo orbit. For $i = 0$, it is obvious. Let us assume that $(\mathfrak{T}_{\sigma_i}(a), a_i) \in A_\epsilon$. If $i$ is even,

$$\left| \frac{a}{6^{\frac{i}{2}}} - a_i \right| < \epsilon,$$

and if $i$ is odd,

$$\left| \frac{a}{2 \times 6^{\frac{(i-1)}{2}}} - a_i \right| < \epsilon.$$

When $i$ is even, $\left|\mathfrak{T}_{\sigma_{i+1}}(a) - a_{i+1}\right| = \left| \frac{a}{2 \times 6^{\frac{(i+1-1)}{2}}} - a_{i+1} \right| \leq \frac{1}{2} \left| \frac{a}{6^{\frac{i}{2}}} - a_i \right| + \left| \frac{a_i}{2} - a_{i+1} \right| < \frac{\epsilon}{2} + \delta = \epsilon$.
Similarly, we can show that $(\mathfrak{T}_{\sigma_{i+1}}(a), a_{i+1}) \in A_\epsilon$ for odd $i$. Hence $\mathfrak{T}$ has *TSP*. □

**Theorem 3.2.** *Let $\mathfrak{T} = \{T; g_\lambda | \lambda \in \Lambda\}$, $\mathfrak{R} = \{T; s_\lambda | \lambda \in \Lambda\}$ be IFSs on a compact uniform space $(T, \mathcal{U})$, in which $\Lambda$ is a non-empty finite set. Let $A \in \mathcal{U}$ be a closed entourage. Suppose $B \in \mathcal{U}$ is another entourage satisfying $(B \circ A)[g_\lambda(a)] \subset s_\lambda(A[a])$ for any $\lambda \in \Lambda$ and for every $a \in T$ then, any $B$-pseudo orbit in $\mathfrak{T}$ can be $A$-shadowed by an orbit in $\mathfrak{R}$.*

*Proof.* Suppose $\{a_i\}_{i \in \mathbb{Z}_+}$ is a $B$-pseudo orbit in $\mathfrak{T}$. Then, $\exists\ \sigma \in \Lambda^{\mathbb{Z}_+}$ such that

$$(g_{\lambda_i}(a_i), a_{i+1}) \in B,\ \forall\ \lambda_i \in \sigma. \tag{3}$$

Define $M_0 = A[a_0],\ M_i = M_{i-1} \cap {\mathfrak{R}_{\sigma_i}}^{-1}(A[a_i])$.
We claim that $\mathfrak{R}_{\sigma_i}(M_i) = A[a_i],\ \forall\ i > 0$. When $i = 1$, by definition of $M_1$, $\mathfrak{R}_{\sigma_1}(M_1) \subset A[a_1]$.
Conversely, let $a \in A[a_1]$. Now, using (3), we have $(g_{\lambda_0}(a_0), a) \in B \circ A$.
Therefore, by hypothesis, $a \in (B \circ A)[g_{\lambda_0}(a_0)] \subset s_{\lambda_0}(A[a_0]) = \mathfrak{R}_{\sigma_1}(A[a_0])$. So, $a = \mathfrak{R}_{\sigma_1}(c)$ for some $c \in A[a_0] \cap {\mathfrak{R}_{\sigma_1}}^{-1}(A[a_1]) = M_1$, which gives that $a \in \mathfrak{R}_{\sigma_1}(M_1)$.
Therefore, $A[a_1] \subset \mathfrak{R}_{\sigma_1}(M_1)$. Hence, $\mathfrak{R}_{\sigma_1}(M_1) = A[a_1]$.
Suppose the proposition holds for $i = n$. Now, for $i = n+1$, by definition of $M_{n+1}$, $\mathfrak{R}_{\sigma_{n+1}}(M_{n+1}) \subset A[a_{n+1}]$. Again, let $b \in A[a_{n+1}]$.
Proceeding as in the case $i = 1$, we get that $b = s_{\lambda_n}(d)$ where $d \in A[a_n] \cap {s_{\lambda_n}}^{-1}(A[a_{n+1}])$.
Now, ${\mathfrak{R}_{\sigma_n}}^{-1}(d) \in {\mathfrak{R}_{\sigma_n}}^{-1}(A[a_n]) \cap {\mathfrak{R}_{\sigma_n}}^{-1}({s_{\lambda_n}}^{-1}(A[a_{n+1}]))$ and it follows that ${\mathfrak{R}_{\sigma_n}}^{-1}(d) \in M_n \cap {\mathfrak{R}_{\sigma_{n+1}}}^{-1}(A[a_{n+1}]) = M_{n+1}$. So, $s_{\lambda_n}(d) \in \mathfrak{R}_{\sigma_{n+1}}(M_{n+1})$.
It follows that $A[a_{n+1}] \subset \mathfrak{R}_{\sigma_{n+1}}(M_{n+1})$. So, $\mathfrak{R}_{\sigma_{n+1}}(M_{n+1}) = A[a_{n+1}]$. Hence, by induction $\mathfrak{R}_{\sigma_i}(M_i) = A[a_i]\ \forall\ i > 0$.
Thus, $\{M_i\}_{i \in \mathbb{Z}_+}$ is a decreasing sequence of non-empty compact sets in $T$. So, $\bigcap_{i \in \mathbb{Z}_+} M_i \neq \phi$. Let $p \in \bigcap_{i \in \mathbb{Z}_+} M_i$.
This implies $p \in M_i,\ \forall\ i \in \mathbb{Z}_+$ which gives $\mathfrak{R}_{\sigma_i}(p) \in \mathfrak{R}_{\sigma_i}(M_i) = A[a_i],\ \forall\ i \in \mathbb{Z}_+$. Thus,

$(\mathfrak{R}_{\sigma_i}(p), a_i) \in A$.
Hence, the $B$-pseudo orbit $\{a_i\}_{i\in\mathbb{Z}_+}$ is $A$-shadowed by some orbit in $\mathfrak{R}$. □

**Corollary 3.1.** *The IFS, $\mathfrak{T} = \{T : g_\lambda|\lambda \in \Lambda\}$ on a compact uniform space $(T,\mathcal{U})$ has TSP if for every closed entourage $A \in \mathcal{U}$ and for any $\lambda \in \Lambda$, there is an entourage $B \in \mathcal{U}$ for which $(B \circ A)[g_\lambda(a)] \subset g_\lambda(A[a])\ \forall\ a \in T$.*

Recently, Ahmadi et al. showed that a dynamical system $(T, g)$ on a uniform space $(T,\mathcal{U})$ has *TSP* iff $(T, g^n)$ has the *TSP* in [2] [**Lemma 3.1**]. We wish to extend the result to *IFS* $\mathfrak{T}$ on a compact uniform space $(T,\mathcal{U})$.

**Theorem 3.3.** *Let $\mathfrak{T}$ be an IFS on a compact uniform space $(T,\mathcal{U})$ with TSP, then for every $n > 1$, $\mathfrak{T}^n$ has TSP.*

*Proof.* Let $A \in \mathcal{U}$. As $\mathfrak{T}$ has *TSP*, there is an entourage $B \in \mathcal{U}$ for which any $B$-pseudo orbit is $A$-shadowed by a point in $T$.
Let $\{a_i\}_{i\in\mathbb{Z}_+}$ be a $B$-pseudo orbit in $\mathfrak{T}^n$, then $\exists\ \sigma = \{\mu_0, \mu_1, \cdots\}$ such that $\forall\ \mu_i \in \sigma,\ (G_{\mu_i}(a_i), a_{i+1}) \in B$ where $G_{\mu_i} = g_{\lambda^i_{n-1}} \circ g_{\lambda^i_{n-2}} \circ \cdots \circ g_{\lambda^i_0}$, $\mu_i = \{\lambda^i_0, \lambda^i_1, \cdots, \lambda^i_{n-1}\} \in \Lambda^n\ \forall\ i \geq 0$.
Let $\{b_j\}_{j\in\mathbb{Z}_+}$ be a sequence defined by $b_j = a_i$ when $j = ni$, and $b_j = g_{\lambda^i_{j-ni-1}} \circ g_{\lambda^i_{j-ni-2}} \circ \cdots \circ g_{\lambda^i_0}(a_i)$, for $ni < j < (i + 1)n$.
Take, $\sigma' = \{\lambda^0_0, \lambda^0_1, \cdots, \lambda^0_{n-1}, \lambda^1_0, \lambda^1_1, \cdots, \lambda^1_{n-1}, \cdots\} \in \Lambda^{\mathbb{Z}_+}$, then $\{b_j\}_{j\in\mathbb{Z}_+}$ is a $B$-pseudo orbit in $\mathfrak{T}$ with respect to $\sigma'$.
If $c \in T$ $A$-shadows this $B$-pseudo orbit, then $(\mathfrak{T}_{\sigma'_j}(c), b_j) \in A,\ \forall\ j \in \mathbb{Z}_+$.
Thus, $(\mathfrak{T}_{\sigma'_{ni}}(c), b_{ni}) \in A,\ \forall\ i \in \mathbb{Z}_+$.
Since $\mathfrak{T}_{\sigma'_{ni}} = g_{\lambda^{i-1}_{n-1}} \circ g_{\lambda^{i-1}_{n-2}} \circ \cdots \circ g_{\lambda^{i-1}_0} \circ \cdots \circ g_{\lambda^0_{n-1}} \circ g_{\lambda^0_{n-2}} \circ \cdots \circ g_{\lambda^0_0} = G_{\mu_{i-1}} \circ \cdots \circ G_{\mu_0} = \mathfrak{T}_{\sigma_i}$, it follows that $\{a_i\}_{i\in\mathbb{Z}_+}$ is a $B$-pseudo orbit in $\mathfrak{T}^n$ which is $A$-shadowed by $c \in T$.
Hence, $\mathfrak{T}^n$ has *TSP*. □

## 4 Topological Chain Properties of Iterated Function Systems

In this section, we study the topological chain properties of iterated function systems on uniform spaces. First, we give definitions of A-chain, topological chain transitivity, topological chain recurrence, totally topological chain transitivity of iterated function systems on uniform spaces. We show that these properties are equivalent in an *IFS* on a connected compact uniform space. We also show that a topologically chain transitive *IFS* on a compact uniform space with topological shadowing property is topologically transitive.

**Definition 4.1.** *Let $\mathfrak{T}$ be an IFS over a uniform space $(T,\mathcal{U})$, and $A \in \mathcal{U}$. A finite sequence $\{a_i\}^n_{i=0}$ in $T$ is said to be an $A$-chain of length $n$ if there is a finite sequence $\{\lambda_0, \lambda_1, \cdots, \lambda_{n-1}\}$ with $\lambda_i \in \Lambda$ satisfying $(g_{\lambda_i}(a_i), a_{i+1}) \in A$ for $i = 0, 1, \cdots, n-1$.*

**Definition 4.2.** *An IFS $\mathfrak{T}$ on a uniform space $(T,\mathcal{U})$ is said to be topological chain transitive$(TCT)$ if for any entourage $A \in \mathcal{U}$ and any two points $a, b \in T$, there exists an $A$-chain from $a$ to $b$. It is said to be totally topological chain transitive$(TTCT)$ if $\mathfrak{T}^n$ is TCT for any $n > 0$.*

**Definition 4.3.** *An IFS $\mathfrak{T}$ on a uniform space $(T,\mathcal{U})$ is said to be topological chain recurrent$(TCR)$ if for any entourage $A \in \mathcal{U}$ and any $a \in T$, there exists an $A$-chain from $a$ to itself.*

**Definition 4.4.** *An IFS $\mathfrak{T}$ on a uniform space $(T,\mathcal{U})$ is called topological chain mixing$(TCM)$ if for any entourage $A \in \mathcal{U}$ and any two points $a, b \in T$, there exists a positive integer $P$ such that for any $n \geq P$, there is an $A$-chain from $a$ to $b$ of length $n$.*

**Theorem 4.1.** *Let $\mathfrak{T} = \{T; g_\lambda | \lambda \in \Lambda\}$ and $\mathfrak{R} = \{R; s_\lambda | \lambda \in \Lambda\}$ be two conjugate IFSs on compact uniform spaces $(T, \mathcal{U})$ and $(R, \mathcal{V})$ respectively. Then $\mathfrak{T}$ is TCT iff $\mathfrak{R}$ does.*

*Proof.* Take $h : T \to R$ to be a topological conjugacy.
Let $A \in \mathcal{V}$ and let $a, b \in R$.
Since $h$ is uniformly continuous, there is an entourage $B \in \mathcal{U}$ such that

$$(u, v) \in B \implies (h(u), h(v)) \in A. \tag{4}$$

First, let us suppose that $\mathfrak{T}$ is *TCT*. Let $\{a_0 = h^{-1}(a), a_1, a_2, \cdots, a_n = h^{-1}(b)\}$ be a $B$-chain from $h^{-1}(a)$ to $h^{-1}(b)$ in $\mathfrak{T}$.
Then, there exists a finite sequence $\{\lambda_0, \lambda_1, \cdots, \lambda_{n-1}\}$ for which $(g_{\lambda_i}(a_i), a_{i+1}) \in B,\ i = 0, 1, \cdots, n-1$.
By (4), $(h(g_{\lambda_i}(a_i)), h(a_{i+1})) \in A$, i.e. $(s_{\lambda_i}(h(a_i)), h(a_{i+1})) \in A$ for $i = 0, 1, \cdots, n-1$.
This implies that $(s_{\lambda_0}(a), h(a_1)) \in A$ and $(s_{\lambda_{n-1}}(h(a_{n-1})), b) \in A$.
Put $b_0 = a, b_n = b$ and $b_i = h(a_i)$ for $i = 1, 2, \cdots, n-1$, then $\{b_0 = a, b_1, \cdots, b_n = b\}$ is an $A$-chain in $\mathfrak{R}$.
Thus, $\mathfrak{R}$ is *TCT*. Similarly, we can prove the converse part.
Hence, $\mathfrak{T}$ is *TCT* iff $\mathfrak{R}$ does. □

**Corollary 4.1.** *Let $\mathfrak{T} = \{T; g_\lambda | \lambda \in \Lambda\}$ and $\mathfrak{R} = \{R; s_\lambda | \lambda \in \Lambda\}$ be two IFSs on compact uniform spaces $(T, \mathcal{U})$ and $(R, \mathcal{V})$ respectively, and $h : T \to R$ a factor map. Then, $\mathfrak{R}$ is TCT if $\mathfrak{T}$ is TCT.*

**Corollary 4.2.** *Let $\mathfrak{T} = \{T; g_\lambda | \lambda \in \Lambda\}$ and $\mathfrak{R} = \{R; s_\lambda | \lambda \in \Lambda\}$ be two conjugate IFSs on compact uniform spaces $(T, \mathcal{U})$ and $(R, \mathcal{V})$ respectively. Then $\mathfrak{T}$ is TCM iff $\mathfrak{R}$ does.*

**Theorem 4.2.** *Let $\mathfrak{T}$ be a TCM IFS on a compact uniform space $(T, \mathcal{U})$, then it is TTCT.*

*Proof.* Fix $n > 0$. Let $A \in \mathcal{U}$, then there is a symmetric entourage $C \in \mathcal{U}$ for which $C^n \subset A$.
Since every finite family of uniformly continuous mappings is uniformly equicontinuous, we can get an entourage $B \in \mathcal{U}$ for which $(u, v) \in B$ implies $(g_{\lambda_{j-1}} \circ g_{\lambda_{j-2}} \circ \cdots \circ g_{\lambda_0}(u), g_{\lambda_{j-1}} \circ g_{\lambda_{j-2}} \circ \cdots \circ g_{\lambda_0}(v)) \in C$ for $j = 1, 2, \cdots, n$ and $\forall\ \lambda_j \in \Lambda$.
Without loss of generality, we take $B \subset C$.
Let $a, b \in T$ be two points.
Since $\mathfrak{T}$ is *TCM*, we can get a $B$-chain from $a$ to $b$ of length $P = np, p \geq 1$, say $\{a = t_0, t_1, \cdots, t_P = b\}$.
Then, there exists a finite sequence $\{\lambda_0, \lambda_1, \cdots, \lambda_{n-1}, \lambda_n, \cdots, \lambda_{np-1}\}$ such that $\forall\ \lambda_i,\ i = 0, 1, \cdots, np-1,\ (g_{\lambda_i}(t_i), t_{i+1}) \in B$.
Now,

$$(g_{\lambda_0}(t_0), t_1) \in B.$$

This gives that

$$(g_{\lambda_{n-1}} \circ \cdots g_{\lambda_1} \circ g_{\lambda_0}(t_0), g_{\lambda_{n-1}} \circ \cdots g_{\lambda_1}(t_1)) \in C.$$

Similarly,

$$(g_{\lambda_{n-1}} \circ \cdots g_{\lambda_1}(t_1), g_{\lambda_{n-1}} \circ \cdots g_{\lambda_2}(t_2)) \in C.$$
$$(g_{\lambda_{n-1}} \circ \cdots g_{\lambda_2}(t_2), g_{\lambda_{n-1}} \circ \cdots g_{\lambda_3}(t_3)) \in C.$$
$$\vdots$$
$$(g_{\lambda_{n-1}} \circ g_{\lambda_{n-2}}(t_{n-2}), g_{\lambda_{n-1}}(t_{n-1})) \in C.$$

Also, $(g_{\lambda_{n-1}}(t_{n-1}), t_n) \in B$.
This implies that $(g_{\lambda_{n-1}}(t_{n-1}), t_n) \in C$.
Thus, $(g_{\lambda_{n-1}} \circ \cdots g_{\lambda_1} \circ g_{\lambda_0}(t_0), t_n) \in C^n \subset A$.
Continuing in this manner, we see that $\{t_0, t_n, t_{2n}, \cdots, t_P\}$ is an $A$-chain from $a$ to $b$ in $\mathfrak{T}^n$.
Hence $\mathfrak{T}$ is *TTCT*. □

The following theorem is an extension of [14], [**Proposition 2.15**] in *IFS* on a compact uniform space.

**Theorem 4.3.** *Let $\mathfrak{T}$ be a TCT IFS on a compact uniform space $(T, \mathcal{U})$ having TSP. Then, it is TT.*

*Proof.* Let $M, N \subset T$ be a pair of non-empty open sets. For a given pair of points $a \in M$ and $b \in N$, suppose $A \in \mathcal{U}$ is a symmetric entourage for which $A[a] \subset M$ and $A[b] \subset N$. Since $\mathfrak{T}$ has *TSP*, there exists an entourage $B \in \mathcal{U}$ such that any $B$-pseudo orbit is $A$-shadowed by a point in $T$. Let $\{a_0 = a, a_1, \cdots, a_n = b\}$ and $\{a_n = b, a_{n+1}, \cdots, a_{n+m} = a\}$ be $B$-chains in $T$ from $a$ to $b$ and $b$ to $a$ respectively. Then, there is a finite sequence $\{\lambda_0, \lambda_1, \cdots, \lambda_{n-1}, \lambda_n, \lambda_{n+1}, \cdots, \lambda_{n+m-1}\}$ such that $(g_{\lambda_i}(a_i), a_{i+1}) \in B$ for $0 \leq i \leq n+m-1$. Define $\{c_j\}_{j \in \mathbb{Z}_+}$ by
$$c_j = \begin{cases} a_j \ , \ j \leq n+m-1 \\ a_{j-k(n+m)} \ , \ (k-1)(n+m) \leq j \leq k(n+m), k \geq 1. \end{cases}$$
Take the sequence $\sigma \in \Lambda^{\mathbb{Z}_+}$ formed by concatenating the finite sequence $\{\lambda_0, \lambda_1, \cdots, \lambda_{n-1}, \lambda_n, \lambda_{n+1}, \cdots, \lambda_{n+m-1}\}$. Then, $\{c_j\}_{j \in \mathbb{Z}_+}$ is a $B$-pseudo orbit with respect to $\sigma$. So, we can find $c \in T$ which $A$-shadows this $B$-pseudo orbit. Therefore, $(c, a) \in A$ and $(\mathfrak{T}_{\sigma_n}(c), b) \in A$, i.e, $c \in A[a]$ and $\mathfrak{T}_{\sigma_n}(c) \in A[b]$. This follows that $\mathfrak{T}_{\sigma_n}(M) \cap N \neq \phi$. Hence, $\mathfrak{T}$ is *TT*. □

Fatehi Nia [15] proved the following theorem as a lemma in iterated function systems. Here, we extend it on uniform spaces. In the proof, we use the fact that every finite family of uniformly continuous functions is uniformly equicontinuous.

**Theorem 4.4.** *Let $\mathfrak{T}$ be an IFS on a compact uniform space $(T, \mathcal{U})$ with TSP. Then $\mathfrak{T}$ is TM if and only if it is TCM.*

*Proof.* First, suppose that $\mathfrak{T}$ is *TM*.
Let $A \in \mathcal{U}$ be an entourage, and $a, b \in T$ a pair of two points.
Since $\{g_\lambda : \lambda \in \Lambda\}$ is a finite family of uniformly continuous functions on $T$, it is uniformly equicontinuous.
Therefore, we can find a symmetric entourage $B \in \mathcal{U}$ such that $B^2 \subset A$ and $(g_\lambda \times g_\lambda)(B) \subset A, \forall \lambda \in \Lambda$.
Since $\mathfrak{T}$ is *TM*, there exist a positive integer $P$ and $\sigma \in \Lambda^{\mathbb{Z}_+}$ such that $\mathfrak{T}_{\sigma_n}(B[a]) \cap B[b] \neq \phi, \forall n \geq P$.
Then, $\exists\, c \in B[a]$ such that $\mathfrak{T}_{\sigma_n}(c) \in B[b]$.
Then $(g_{\lambda_0}(a), g_{\lambda_0}(c)) \in A$ and $\mathfrak{T}_{\sigma_n}(c) \in A[b]$.
This follows that $(g_{\lambda_0}(a), \mathfrak{T}_{\sigma_1}(c)) \in A$ and $\mathfrak{T}_{\sigma_n}(c) \in A[b]$.
Therefore, $\{c_0 = a, c_1 = \mathfrak{T}_{\sigma_1}(c), \cdots, c_{n-1} = \mathfrak{T}_{\sigma_{n-1}}(c), b\}$ is an $A$-chain of length $n$.
Hence, $\mathfrak{T}$ is *TCM*.

Conversely, let $\mathfrak{T}$ be *TCM*.
Let $M, N$ be any pair of non-empty open sets in $T$.
For any pair of points $a \in M$ and $b \in N$, let $A \in \mathcal{U}$ be a symmetric entourage for which $A[a] \subset M$ and $A[b] \subset N$.
Since, $\mathfrak{T}$ has *TSP*, there exists an entourage $B \in \mathcal{U}$ such that any $B$-pseudo orbit is $A$-shadowed by a point in $T$.
Again, since $\mathfrak{T}$ is *TCM* and $a, b \in T$, there exists a positive integer $P$ such that $\forall\ n \geq P$, there is a $B$-chain from $a$ to $b$ of length $n$.
Let $\{a = a_0, a_1, a_2, \cdots, a_{n-1}, a_n = b\}$ be a $B$-chain from $a$ to $b$.
Then, there exists a finite sequence $\{\lambda_0, \lambda_1, \cdots, \lambda_{n-1}\}$ such that

$$(g_{\lambda_i}(a_i), a_{i+1}) \in B,$$

for all $i = 0, 1, \cdots, n-1$
Put $\sigma = \{\lambda_0, \lambda_1, \cdots, \lambda_{n-1}, \lambda_0, \lambda_1, \cdots, \lambda_{n-1}, \cdots\}$ formed by concatenating the finite sequence

$$\{\lambda_0, \lambda_1, \cdots, \lambda_{n-1}\},$$

then $\{a_0 = a, a_1, \cdots, a_n = b, a_{n+1} = g_{\lambda_0}(b), a_{n+2} = g_{\lambda_1} \circ g_{\lambda_0}(b), \cdots, a_{n+m} = \mathfrak{T}_{\sigma_m}(b), \cdots\}$ is a $B$-pseudo orbit in $T$.
If $c \in T$ $A$-shadows this $B$-pseudo orbit, then $(c, a) \in A$ and $(\mathfrak{T}_{\sigma_n}(c), b) \in A$, i.e, $c \in A[a]$ and $\mathfrak{T}_{\sigma_n}(c) \in A[b]$.
It follows that $\mathfrak{T}_{\sigma_n}(M) \cap N \neq \phi$, $\forall\ n \geq P$.
Hence, $\mathfrak{T}$ is *TM*. □

In [16], Richeson and Wiseman proved that the chain mixing, totally chain transitive, chain transitive, and chain recurrent properties of a dynamical system $(T, g)$ are equivalent over a connected metric space $T$. Mehdi [15] extended this equivalence on iterated function systems. Recently, Ahmadi et al. [2] proved that the *TCM, TTCT, TCT*, and *TCR* properties of a dynamical system over a connected compact uniform space are equivalent. Here, in Theorem 4.7, we prove that the *TCM, TTCT, TCT, and TCR* properties for *IFS* are equivalent over a connected compact uniform space.

**Lemma 4.1.** *Let $\mathfrak{T}$ be a TCT iterated function system on a compact uniform space $(T, \mathcal{U})$ and $A \in \mathcal{U}$. Then, $\exists\ p_A \in \mathbb{N}$ such that for $a \in T$, $p_A$ is the greatest common divisor(gcd) of the lengths of all $A$-chains from $a$ to itself.*

*Proof.* Let $b \in T$ be fixed and $p_A(b)$ be the gcd of all $A$-chains from $b$ to itself. Let $a \in T$ be a point and $\{a_0 = a, a_1, \cdots, a_n = a\}$ an $A$-chain from $a$ to itself of length $n$. Then, there exists a finite sequence $\{\lambda_0, \lambda_1, \cdots, \lambda_{n-1}\}$ for which $(g_{\lambda_i}(a_i), a_{i+1}) \in A$, $\forall\ i = 0, 1, \cdots, n-1$. Since $\mathfrak{T}$ is *TCT* and $b, a \in T$, $\exists$ $A$-chains from $b$ to $a$ and from $a$ to $b$ say, $\{b_0 = b, b_1, \cdots, b_m = a\}$ and $\{c_0 = a, c_1, \cdots, c_k = b\}$ respectively. Thus, there exist finite sequences $\{\lambda_0^{'}, \lambda_1^{'}, \cdots, \lambda_{m-1}^{'}\}$ and $\{\lambda_0^{''}, \lambda_1^{''}, \cdots, \lambda_{k-1}^{''}\}$ such that $(g_{\lambda_j^{'}}(a_j), a_{j+1}) \in A$ and $(g_{\lambda_l^{''}}(a_l), a_{l+1}) \in A$, $\forall\ j = 0, 1, \cdots, m-1$ and $\forall\ l = 0, 1, \cdots, k-1$ respectively. Consider the finite sequence $\{\lambda_0^{'}, \lambda_1^{'}, \cdots, \lambda_{m-1}^{'}, \lambda_0^{''}, \lambda_1^{''}, \cdots, \lambda_{k-1}^{''}\}$, then $\{b_0 = b, b_1, \cdots, b_m = a = c_0, c_1, \cdots, c_k = b\}$ is an $A$-chain from $b$ to itself. Therefore, $m + k$ is a multiple of $p_A(b)$. Similarly, $\{b_0 = b, b_1, \cdots, b_m = a = a_0, a_1, \cdots, a_n = a = c_0, c_1, \cdots, c_k = b\}$ is also an $A$-chain from $b$ to itself. Thus, $m + n + k$ is also a multiple of $p_A(b)$. It implies that $n$ is a multiple of $p_A(b)$. Hence, for $a \in T$, $p_A = p_A(b) = p_A(a)$ is the gcd of the lengths of all $A$-chains from $a$ to itself. □

**Lemma 4.2** ([4], Theorem 1.0.1)**.** *Let $a_1, a_2, \cdots, a_n \in \mathbb{N}$. If $(a_1, \cdots, a_n) = 1$, then there is $P \in \mathbb{N}$ so that any integer $s \geq P$ is representable as a non-negative combination of $a_1, a_2, \cdots, a_n$.*

Let $\mathfrak{T}$ be a *TCT IFS* on a uniform space $(T,\mathcal{U})$ and $A \in \mathcal{U}$. Let $\sim_A$ be a relation on $T$ defined by $a \sim_A b$ iff there exists an $A$-chain from $a$ to $b$ of length a multiple of $p_A$. Then, it is true that $\sim_A$ is an equivalence relation.

**Claim 1:** Suppose $a \sim_A b$, then the length of every $A$-chain from $a$ to $b$ is a multiple of $p_A$.
**Proof:** Let $a \sim_A b$. As $\sim_A$ is an equivalence relation, $b \sim_A a$. Therefore, $\exists$ an $A$-chain say $\{b_0 = b, b_1, \cdots, b_n = a\}$ from $b$ to $a$, where $n$ is a multiple of $p_A$. Thus, there is a finite sequence $\{\lambda_0, \lambda_1, \cdots, \lambda_{n-1}\}$ for which $(g_{\lambda_i}(b_i), b_{i+1}) \in A,\ \forall\ i = 0, 1, \cdots, n-1$. Let $\{a_0 = a, a_1, \cdots, a_m = b\}$ be any $A$-chain from $a$ to $b$. Then, we can get a finite sequence $\{\lambda_0^{'}, \lambda_1^{'}, \cdots, \lambda_{m-1}^{'}\}$ so that $(g_{\lambda_j^{'}}(a_j), a_{j+1}) \in A,\ \forall\ j = 0, 1, \cdots, m-1$. Consider the finite sequence $\{\lambda_0, \lambda_1, \cdots, \lambda_{n-1}, \lambda_1^{'}, \cdots, \lambda_{m-1}^{'}\}$, then $\{b_0 = b, b_1, \cdots, b_n = a = a_0, a_1, \cdots, a_m = b\}$ is an $A$-chain from $b$ to itself of length $n+m$. It implies that, $n+m$ is a multiple of $p_A$, therefore $m$ is a multiple of $p_A$. Hence, the length of every $A$-chain from $a$ to $b$ is a multiple of $p_A$.

**Claim 2:** Suppose $[a]_A$ is the equivalence class of $a$ under $\sim_A$, then there exists exactly $p_A$ equivalence classes under $\sim_A$ and $g_{\lambda_{p_A-1}} \circ g_{\lambda_{p_A-2}} \circ \cdots \circ g_{\lambda_0}([a]_A) = [a]_A,\ \forall\ \lambda_i \in \Lambda$.
**Proof:** Suppose $\{a_0 = g_{\lambda_{p_A-1}} \circ g_{\lambda_{p_A-2}} \circ \cdots \circ g_{\lambda_0}(a), a_1, \cdots, a_n = a\}$ is an $A$-chain from $g_{\lambda_{p_A-1}} \circ g_{\lambda_{p_A-2}} \circ \cdots \circ g_{\lambda_0}(a)$ to $a$ of length $n$. Again, $\{a, g_{\lambda_0}(a), g_{\lambda_1} \circ g_{\lambda_0}(a), \cdots, g_{\lambda_{p_A-1}} \circ g_{\lambda_{p_A-2}} \circ \cdots \circ g_{\lambda_0}(a)\}$ is an $A$-chain of length $p_A$ from $a$ to $g_{\lambda_{p_A-1}} \circ g_{\lambda_{p_A-2}} \circ \cdots \circ g_{\lambda_0}(a)$. Then, $\{a, g_{\lambda_0}(a), g_{\lambda_1} \circ g_{\lambda_0}(a), \cdots, g_{\lambda_{p_A-1}} \circ g_{\lambda_{p_A-2}} \circ \cdots \circ g_{\lambda_0}(a), a_1, \cdots, a_n = a\}$ is an $A$-chain from $a$ to itself of length $n+p_A$ which implies that $n$ is a multiple of $p_A$. Thus, $g_{\lambda_{p_A-1}} \circ g_{\lambda_{p_A-2}} \circ \cdots \circ g_{\lambda_0}(a) \in [a]_A$. Therefore, $[g_{\lambda_{p_A-1}} \circ g_{\lambda_{p_A-2}} \circ \cdots \circ g_{\lambda_0}(a)]_A = [a]_A$. Also, for every $0 \le i < j < p_A$, one can easily show that $g_{\lambda_{i-1}} \circ g_{\lambda_{i-2}} \circ \cdots \circ g_{\lambda_0}(x) \nsim_A g_{\lambda_{j-1}} \circ g_{\lambda_{j-2}} \circ \cdots \circ g_{\lambda_0}(a)$. Hence, one can easily show that there exists exactly $p_A$ equivalence classes under $\sim_A$ namely, $[a]_A,\ [g_{\lambda_0}(a)]_A,\ [g_{\lambda_0} \circ g_{\lambda_1}(a)]_A, \cdots,\ [g_{\lambda_{p_A-2}} \circ \cdots \circ g_{\lambda_0}(a)]_A$. Clearly, $g_{\lambda_{p_A-1}} \circ g_{\lambda_{p_A-2}} \circ \cdots \circ g_{\lambda_0}([a]_A) = [a]_A$.

**Theorem 4.5.** *Let $\mathfrak{T}$ be a TCT IFS on a compact uniform space $(T,\mathcal{U})$. Then, for any entourage $A \in \mathcal{U}$ and any $a \in T$, $[a]_A$ is both open and closed.*

*Proof.* Let $A \in \mathcal{U}$. Then, we can get a symmetric entourage $B \in \mathcal{U}$ so that $B^2 \subset A$. Let $a, b \in T$ be two points such that $b \in [a]_A$. Since $\mathfrak{T}$ is *TCT*, there is a $B$-chain say $\{a_0 = a, a_1, \cdots, a_n = b\}$ from $a$ to $b$ of length $n$. Thus, there exists a finite sequence $\{\lambda_0, \lambda_1, \cdots, \lambda_{n-1}\}$ so that $(g_{\lambda_i}(a_i), a_{i+1}) \in B,\ i = 0, 1, \cdots, n-1$. Also, $(a_{i+1}, a_{i+1}) \in B$. So, $(g_{\lambda_i}(a_i), a_{i+1}) \in B^2 \subset A$. Therefore, $\{a_0 = a, a_1, \cdots, a_n = b\}$ is an $A$-chain from $a$ to $b$. Hence, by Claim 1, $n$ is a multiple of $p_A$. Now, for any $c \in B[b],\ (b, c) \in B$. As $\{a_0 = a, a_1, \cdots, a_n = b\}$ is a $B$-chain, $(g_{\lambda_{n-1}}(a_{n-1}), b) \in B$. Therefore, $(g_{\lambda_{n-1}}(a_{n-1}), c) \in B^2 \subset A$. This implies that $\{a_0 = a, a_1, \cdots, a_n = c\}$ is an $A$-chain from $a$ to $c$ whose length is a multiple of $p_A$. Thus, $c \sim_A a$ which follows that $c \in [a]_A$. It implies that $c \in B[b] \subset [a]_A$. Hence $[a]_A$ is open.

Next, let $b \in \overline{[a]_A}$. Then, $B[b] \cap [a]_A \neq \phi$. Therefore, there exists some $c \in B[b] \cap [a]_A$. Since $\mathfrak{T}$ is *TCT*, there exists a $B$-chain say $\{c_0 = a, c_1, c_2, \cdots, c_m = c\}$ from $a$ to $c$. Since $c \in B[b]$, by similar argument as above, $\{c_0 = a, c_1, c_2, \cdots, c_m = b\}$ is an $A$-chain from $a$ to $b$ whose length is a multiple of $p_A$. Therefore, $a \sim_A b$ which implies that $b \in [a]_A$. This implies that $[a]_A \subset \overline{[a]_A} \subset [a]_A$. Hence, $[a]_A$ is closed. □

**Lemma 4.3** ([13], Proposition 8.16)**.** *Let $\mathfrak{A}$ be an open covering of a compact uniform space $(T,\mathcal{U})$. There exists an entourage $B \in \mathcal{U}$ such that $\mathcal{C}(B) = \{B[a] \mid a \in T\}$ refines $\mathfrak{A}$, i.e., each of the uniform neighbourhoods $B[a]$ is contained in some member of $\mathfrak{A}$.*

**Proposition 4.1.** *Let $\mathfrak{T}$ be a TCT IFS on a compact uniform space $(T,\mathcal{U})$ and $A \in \mathcal{U}$. If $\mathfrak{T}^{p_A}$ is TCT, then $p_A = 1$.*

*Proof.* If possible, let $p_A > 1$. Let $T = \bigcup_{l=1}^{p_A} M_l$, where $M_l$ are equivalence classes under $\sim_A$. As $M_l$s are open, the family $\mathfrak{A} = \{M_1, M_2, \cdots, M_{p_A}\}$ is an open cover of $T$. By above Lemma 4.3, there exists a symmetric entourage $B$ such that $\mathcal{C}(B) = \{B[a] \mid a \in T\}$ refines $\mathfrak{A}$. Without loss of generality, let us assume that $B \subset A$. Let $a \in M_p$ and $b \in M_q$ for $p \neq q$. Since $\mathfrak{T}^{p_A}$ is TCT, there is a $B$-chain say $\{a_0 = a, a_1, \cdots, a_n = b\}$ from $a$ to $b$ in $\mathfrak{T}^{p_A}$. Thus, there exists a finite sequence $\{\mu_0, \mu_1, \cdots, \mu_{n-1}\}$ such that $(G_{\mu_i}(a_i), a_{i+1}) \in B,\ i = 0, 1, \cdots, n-1$, where, $\mu_i = \{\lambda_0^i, \lambda_1^i, \cdots, \lambda_{p_A-1}^i\}$ and $G_{\mu_i}(a_i) = g_{\lambda_{p_A-1}^i} \circ \cdots g_{\lambda_1^i} \circ g_{\lambda_0^i}(a_i)$. Then, $(g_{\lambda_{p_A-1}^0} \circ \cdots g_{\lambda_1^0} \circ g_{\lambda_0^0}(a_0), a_1) \in B \subset A$. Therefore, when $i = 0$, $\{a_0, g_{\lambda_0^0}(a_0), \cdots, g_{\lambda_{p_A-2}^0} \circ \cdots g_{\lambda_1^0} \circ g_{\lambda_0^0}(a_0), a_1\}$ is an $A$-chain of length $p_A$ in $\mathfrak{T}$ corresponding to the finite sequence $\{\lambda_0^0, \lambda_1^0, \cdots, \lambda_{p_A-1}^0\}$. This implies that $a_1 \in [a_0]_A = M_p$ as $a_0 = a \in M_p$. Thus, by induction on $i$, we get that $b = a_n \in M_p$. This is a contradiction. Hence, $p_A = 1$. □

**Lemma 4.4.** *Let $\mathfrak{T}$ be a TCT IFS on a compact uniform space $(T,\mathcal{U})$ and $A \in \mathcal{U}$. If $p_A = 1$, then for any $a \in T$, there are two $A$-chains from $a$ to itself which have relatively prime lengths.*

The proof is similar to the proof of **Lemma 3.5** of [2].

**Lemma 4.5.** *Let $\mathfrak{T}$ be an IFS on a compact uniform space $(T,\mathcal{U})$. If $\mathfrak{T}$ is TCT, then for any entourage $A \in \mathcal{U}$, there exists $K \in \mathbb{N}$ so that for any $a, b \in T$, there is an $A$-chain from $a$ to $b$ of length less than or equal to $K$.*

*Proof.* Let $A \in \mathcal{U}$ be an entourage, and $B \in \mathcal{U}$ a symmetric entourage such that $B^2 \subset A$. As T is compact, there are points $a_1, a_2, \cdots, a_n \in T$ for which $T = \bigcup_{l=1}^n B[a_l]$. Since $\mathfrak{T}$ is TCT, for $i, j \in \{1, 2, \cdots, n\}$, there is a $B$-chain from $a_i$ to $a_j$ of length $k_{i,j}$. Let $K =$max$\{k_{i,j} + 1 \mid i, j \in \{1, 2, \cdots, n\}\}$. Let $a, b \in T$. There are $p, q \in \{1, 2, \cdots, n\}$ such that $g_{\lambda_0}(a) \in B[a_p]$ and $b \in B[a_q]$. Let $\{a_p = c_0, c_1, \cdots, c_{k_{p,q}} = a_q\}$ be a $B$-chain from $a_p$ to $a_q$ of length $k_{p,q}$. Then, there exists a finite sequence $\{\lambda_0, \lambda_1, \cdots, \lambda_{k_{p,q}-1}\}$ such that $(g_{\lambda_i}(c_i), c_{i+1}) \in B,\ \forall\ i = 0, 1, \cdots, k_{p,q} - 1$. Also, $(c_{i+1}, c_{i+1}) \in B$. So, $(g_{\lambda_i}(c_i), c_{i+1}) \in B^2 \subset A$. Thus, $\{a_p = c_0, c_1, \cdots, c_{k_{p,q}} = a_q\}$ is an $A$-chain of length $k_{p,q}$. Since $b \in B[a_q]$, $(a_q, b) \in B$ which implies that $(g_{\lambda_{k_{p,q}-1}}(c_{k_{p,q}-1}), b) \in A$. Now, $g_{\lambda_0}(a) \in B[a_p] \subset A[a_p]$. Therefore, for $\{\lambda_0, \lambda_0, \lambda_1, \lambda_2, \cdots, \lambda_{k_{p,q}-1}\}$, the finite sequence $\{a, a_p = c_0, c_1, \cdots, c_{k_{p,q}-1}, b\}$ is an $A$-chain from $a$ to $b$ of length less than or equal to $K$. □

**Theorem 4.6.** *Let $\mathfrak{T}$ be a TCT IFS on a connected compact uniform space $(T,\mathcal{U})$, then it is TCM.*

*Proof.* For any entourage $A \in \mathcal{U}$, $p_A = 1$ for if $p_A > 1$, then $T = \bigcup_{l=1}^{p_A} M_l$, a disjoint union of clopen subsets of $T$, which is a contradiction as $T$ is connected. Let $a \in T$. Since $p_A = 1$, by Lemma 4.4, there are two $A$-chains from $a$ to itself which have relatively prime lengths. Concatenating these two chains and using Lemma 4.2, there is a positive integer $P$ so that for any $s \geq P$, we can get an $A$-chain of length $s$ from $a$ to itself. Let $b \in T$ be a point. Then by Lemma 4.5, $\exists\ K \in \mathbb{N}$ such that there is an $A$-chain from $a$ to $b$ of length less than or equal to $K$. Thus, by joining the $A$-chain from $a$ to itself to the chain from $a$ to $b$, we can find an $A$-chain from $a$ to $b$ of length greater than $K + P$. Thus, $\mathfrak{T}$ is TCM. □

In the following theorem, we use the concept of chain equivalence. For a uniform space $(T,\mathcal{U})$, we say that $a$ and $b$ are chain equivalent if for every entourage $A \in \mathcal{U}$ there are $A$-chains from $a$ to $b$ and from $b$ to $a$.

**Theorem 4.7.** *Let $\mathfrak{T}$ be an IFS on a connected compact uniform space $(T,\mathcal{U})$. Then the following are equivalent:*
*(1) $\mathfrak{T}$ is TCM.*
*(2) $\mathfrak{T}$ is TTCT.*
*(3) $\mathfrak{T}$ is TCT.*
*(4) $\mathfrak{T}$ is TCR.*

*Proof.* By Theorem 4.2 $(1) \implies (2)$. And $(2) \implies (3) \implies (4)$ follows from the definitions. So, it is clear that $(1) \implies (2) \implies (3) \implies (4)$. Again, by Theorem 4.6, $(3) \implies (1)$. Thus, it suffices to show that $(4) \implies (3)$.
Suppose $(4)$ holds true. It is obvious that the chain equivalent relation is an equivalence relation. Therefore $T = \bigcup_{a\in T} [a]$, where $[a]$ is the equivalence class of $a$. Let $A \in \mathcal{U}$ be an entourage, and $B$ a symmetric entourage such that $B^2 \subset A$. By uniform equicontinuity, one can find a symmetric entourage $C \in \mathcal{U}$ such that $C^2 \subset B$ and

$$(g_\lambda \times g_\lambda)(C) \subset B, \; \forall \; \lambda \in \Lambda. \tag{5}$$

We claim that $[a]$ is open. Let $b \in [a]$ be an arbitrary point. Take an arbitrary point $c \in C[b]$. Let $\{b_0 = b, b_1, b_2, \cdots, b_n = b\}$ be a $C$-chain from $b$ to itself. Thus, there is a finite sequence $\{\lambda_0, \lambda_1, \cdots, \lambda_{n-1}\}$ so that

$$(g_{\lambda_i}(b_i), b_{i+1}) \in C, \; i = 0, 1, \cdots, n-1. \tag{6}$$

By the inclusions

$$C \subset C^2 \subset B \subset B^2 \subset A,$$

$\{b_0 = b, b_1, b_2, \cdots, b_n = b\}$ is both $B$-chain and $A$-chain from $b$ to itself. Let $\{a = a_0, a_1, \cdots, a_m = b\}$ be an $A$-chain from $a$ to $b$. Then, there is a finite sequence $\{\lambda_0^{'}, \lambda_1^{'}, \cdots, \lambda_{m-1}^{'}\}$ such that $(g_{\lambda_j^{'}}(a_j), a_{j+1}) \in A$ for $j = 0, 1, \cdots, m-1$. Now, $(g_{\lambda_{n-1}}(b_{n-1}), b) \in B$ and $(b, c) \in C \subset B$. It follows that $(g_{\lambda_{n-1}}(b_{n-1}), c) \in B^2 \subset A$. Therefore, $\{b_0 = b, b_1, b_2, \cdots, b_{n-1}, c\}$ is an $A$-chain from $b$ to $c$. Consider the finite sequence $\{\lambda_0^{'}, \lambda_1^{'}, \cdots, \lambda_{m-1}^{'}, \lambda_0, \lambda_1, \cdots, \lambda_{n-1}\}$, then $\{a = a_0, a_1, \cdots, a_m = b = b_0, b_1, b_2, \cdots, b_{n-1}, c\}$ is an $A$-chain from $a$ to $c$.
Next, take an $A$-chain $\{c_0 = b, c_1, \cdots, c_k = a\}$ from $b$ to $a$ with associated finite sequence $\{\lambda_0^{''}, \lambda_1^{''}, \cdots, \lambda_{k-1}^{''}\}$, i.e. $(g_{\lambda_l^{''}}(c_l), c_{l+1}) \in A$ for $l = 0, 1, \cdots, k-1$. Now, $c \in C[b], \; (c, b) \in C$ so, using (5), we have $(g_{\lambda_0}(c), g_{\lambda_0}(b)) \in B$. By (6), $(g_{\lambda_0}(b), b_1) \in B$. It follows that $(g_{\lambda_0}(c), b_1) \in B^2 \subset A$. Therefore $\{c, b_1, \cdots, b_n = b = c_0, c_1, \cdots, c_k = a\}$ is an $A$-chain from $c$ to $a$ with associated finite sequence $\{\lambda_0, \lambda_1, \cdots, \lambda_{n-1}, \lambda_0^{''}, \lambda_1^{''}, \cdots, \lambda_{k-1}^{''}\}$. It follows that $a$ and $c$ are chain equivalent. Therefore, $c \in [a]$. As $c$ was chosen arbitrarily from $C[b]$, we have $C[b] \subset [a]$. Thus, for every $b \in [a]$, we have $C[b] \subset [a]$. This shows that $[a]$ is open. Since $T$ is connected, we have $T = [a]$. Hence $\mathfrak{T}$ is *TCT*. □

## 5 Conclusions

It has been found that the *TSP*, *TCT* and *TCM* of an *IFS* are invariant under a topological conjugacy on compact uniform spaces. It has been shown that if $\mathfrak{T}$ is an *IFS* on a compact uniform space $(T,\mathcal{U})$ with *TSP*, then for any $n > 1$, $\mathfrak{T}^n$ has *TSP*. Also, it has been proved that if $\mathfrak{T}$ is an *IFS* on a compact uniform space $(T,\mathcal{U})$ with *TSP*, then $\mathfrak{T}$ is *TM* if and only if it is *TCM*. It has been proved that *TCM*, *TTCT*, *TCT* and *TCR* properties are equivalent in an *IFS* on a connected compact uniform space. We have extended various standard notions of iterated function systems to the case when the phase space is a compact uniform space. The class of compact metric spaces is a subclass of the class of compact uniform spaces, so the research work will generate interests of wider applications.

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